Supplementary material to the paper:
"The Marginal Distribution Function of Threshold-type Processes with Central Symmetric Innovations"
Statistics (2022), 56.

1) Description of the Matlab files:

- **pdf_SETAR211_Normal.m**: Computes the exact values of the mean, variance, skewness, kurtosis, and first-lag autocorrelation function of the SETAR$(2;1,1)$ process (31) with $\mathcal{N}(0,1)$ innovations; see Table 1.
- **pdf_SETAR_Laplace.m**: Computes the exact values of the mean, variance, skewness, kurtosis, and first-lag autocorrelation function of the SETAR$(2;1,1)$ process (52) with $\mathcal{L}(0,1)$ innovations; see Table 3.
- **pdf_TARMA21111.m**: Computes approximate values of the mean, variance, skewness, and kurtosis of various SETARMA$(2;1,1,1,1)$ processes with $\mathcal{N}(0,1)$, $\mathcal{L}(0,1)$, $\mathcal{C}(0,1)$, and Student t innovations. Used for $\mathcal{L}(0,1)$ results in Table 4. The exact values in Table 4 are computed with **pdf_SETAR_Laplace.m**.
- **pdf_asMA11_Normal.m**: Computes the exact pdf and moments of an asMA$(1,1)$ process with $\mathcal{N}(0,1)$ innovations; see Remark 6.3.
- **pdf_SETAR211_Cauchy_2versions.m**: Computes two versions of the pdf of a SETAR$(2;1,1)$ process with parameter $\alpha=-\beta$ and $\mathcal{C}(0,1)$ innovations; see Section 6.3.
- **Moments_SETAR200_Gauss.m**: Computes the exact moments and ACF of a SETAR$(2;0,0)$ process (also called PCM) with $\mathcal{N}(0,1)$ innovations; see Example 3.2 (and Figure 1).

2) Miscellanea:

- The following ML codes can be used to replicate the results given by
  Hervé, L. and Ledoux, J. (2020). State-discretization of V-geometrically ergodic Markov chains and convergence to the stationary distribution. *Methodology and Computing in Applied Probability*, 22: 905–925.

  pdf_ARCH_Gaussian_discretization.m

  pdf_Exponential_discretization.m

  pdf_Uniform_discretization.m

  pdf_Gaussian_WN_discretization.m

- The following R-codes can be used to replicate the results given by
  Li, D. and Qiu, J. (2020). The marginal density of a TMA(1) process. *Journal of Time Series Analysis*, 41, 476–484.

  Note, in all cases the R-codes call the package: matlab (for "meshgrid").

AbsoluteAR.R	**Bilinear.R**	**CorrectTMA.R**
DAR.R	**EXPAR.R**	

- The following ML code is based on the paper by Guay, A. and Scaillet, O. (2003). Indirect inference, nuisance parameter, and threshold moving average models. *Journal of Business & Economic Statistics*, 21: 122–132.

  Skew_Kurt_CasMA_Normal.m

3) Discretization of SETAR$(2;1,1)$

In this supplement, we discuss a numerical method for getting the limiting stationary distribution directly, i.e., without any recursive procedure. Let $\{Y_t, t \in \mathbb{N}\}$ be a V-geometrically ergodic Markov chain (see, e.g., Meyn and Tweedie (1993)) on a measurable space $\mathbb{Y}$ with invariant probability density function (pdf) π, and with transition probability matrix $\mathbf{P} = \{p(x,y), x, y \in \mathbb{Y}\}$. Recently, Hervé and Ledoux (2020) proposed a so-called state-discretization scheme providing a computable sequence of probabilities $\{\widehat{\pi}_m\}_{m\geq 1}$ on $\mathbb{Y}$ which approximates π as $m \to \infty$. To obtain $\widehat{\pi}_m$ $(m = 1, 2, \ldots)$, a stochastic matrix $\mathbf{B}_m$ is constructed from $\mathbf{P}$ by discretizing the kernel transition probabilities $p(x,y)$. Then $\widehat{\pi}_m$ is defined as the left $\mathbf{B}_m$-invariant probability vector. The associated stationary pdf $\mathfrak{p}$ is computed as the absolutely continuous part $\mathfrak{p}_m$ of $\widehat{\pi}_m$ w.r.t. a positive σ-additive measure on the state space of the Markov chain. The authors prove that the sequence $\{\mathfrak{p}_m\}_{m\geq 1}$ converges point-wise to $\mathfrak{p}$. For a SETAR$(2;1,1)$ process $\{Y_t, t \in \mathbb{Z}\}$ with parameters α_1^- and α_1^+ and innovations following a central symmetric distribution function $\mathcal{G}_{(\cdot)}$, the discretization scheme goes as follows.

(i) Fix an appropriate positive integer m such that $\mathbb{Y}_m := [-m, m[$ and choose $m^- = -m$ and $m^+ = m$ such that $[m^-, m^+[\subset \mathbb{Y}_m$.

(ii) Choose a mesh δ_m $(\ll 1)$ of the partition of $\mathbb{Y}_m$ such that $q_{\max} = (m^+ - m^-)/\delta_m \in \mathbb{Z}^+$. Construct the set of equally spaced grid values $\{x_{i,m} := m^- + i\delta_m\}$ $(i = 0, \ldots, q_{\max})$ and consider the finite partition $\mathbb{Y}_{i,m} = [x_{i,m}, x_{i+1,m}[$ $(i = 0, \ldots, q_{\max} - 1)$ of $[m^-, m^+[$.

(iii) Initialize the $(q_{max}+1) \times (q_{max}+1)$ matrix $\mathbf{B}_m$. Next, for $i, j = 0, \ldots, q_{\max} - 1$, compute

$$p_{i,m}(x) := \min_{v \in \mathbb{Y}_{i,m}} \{\mathcal{G}_{(v-\alpha_1^- x_{i,m})}, \mathcal{G}_{(v-\alpha_1^- x_{i+1,m})}\} \mathbb{I}_{(-\infty, x_{i-1,m}]}(v) + \min_{v \in \mathbb{Y}_{i,m}} \{\mathcal{G}_{(v-\alpha_1^+ x_{i,m})}, \mathcal{G}_{(v-\alpha_1^+ x_{i+1,m})}\} \mathbb{I}_{(x_{i-1,m}, \infty)}(v), \tag{1a}$$

$$b_{ij,m} := \int_{x_{j,m}}^{x_{j+1,m}} p_{i,m}(x) \mathrm{d}x. \tag{1b}$$

(iv) Let $j_0 = \lfloor q_{\max}/2 \rfloor$ with $\lfloor \cdot \rfloor$ the integer part function on $\mathbb{R}$. Then for $j \equiv j_0$ and $i = 0, \ldots, q_{\max} - 1$, compute

$$b_{i,j_0,m} := b_{i,j_0,m} + 1 - \int_{m^-}^{m^+} p_{i,m}(x) \mathrm{d}x \quad \text{with } b_{q_{\max},j_0,m} := 1.$$

(v) Compute the invariant probability vector $\boldsymbol{\pi}_m = (\pi_{0,m}, \ldots, \pi_{q_{max}-1,m}, 0)$ of $\mathbf{B}_m$.

(vi) Finally, the non-negative density function $\mathfrak{p}_m(\cdot)$ is defined by

$$\mathfrak{p}_m(x) := \mathbb{I}_{[m^-, m^+[}(x) \sum_{i=0}^{q_{\max}-1} \pi_{i,m} p_{i,m}(x), \quad \forall x \in \mathbb{R}.$$

Remark 1: Observe that (1b) involves the computation of the function $v \mapsto p(v,x)$ on a small interval of length δ_m. This should not be computationally expensive, provided the pdf under consideration has no local minima. In addition, it is good to mention that the results of the approximation in step (vi) are sensitive to the support $\mathbb{Y}_m$. A sufficiently large support interval $[m^-, m^+[$ that covers twice the range of a sufficient long sample series generated from the model is recommended.

Remark 2: Similar to the multiple-regime PCM in Section 4 of De Gooijer (2022), the above algorithm can be generalized to a SETAR$(k;1,1)$ $(k \geq 3)$ model by extending the expression on the right-hand side of (1a) in an obvious way.

Example: A SETAR$(2; 1, 1)$ process with parameters of opposite signs and the same absolute value, with $r = 0$, and with $\{\varepsilon_t\} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$ is given by

$$Y_t = \alpha_1^- Y_{t-1} \mathbb{I}_{(-\infty,0]}(Y_{t-1}) + \alpha_1^+ Y_{t-1} \mathbb{I}_{(0,\infty)}(Y_{t-1}) + \varepsilon_t, \quad \alpha_1^+ = -\alpha_1^- \equiv \alpha,\ (0 < \alpha < 1), \qquad (2)$$

where $\mathbb{I}_A(\cdot)$ denotes the indicator function of a set A. For parameters $\alpha = 0.5$ and $\alpha = 0.8$, we noted that the graphs of the pdfs $\mathfrak{p}_m(x)$ are so near the exact stationary distribution that in both cases it appears that only one curve is plotted. This does not occur for $\alpha = 0.9$.

The values of the approximate central moments can be summarized as follows:

- With $m^- = -m^+ = -4$ and $q_{\max} = 4,000$, the implied mean, variance, skewness, and kurtosis of the stationary distribution for $\alpha = 0.5$ are, respectively, 0.44 (μ), 1.01 (σ^2), 0.03 ($\mathcal{S}$), and 2.98 ($\mathcal{K}$).
- With $m^- = -m^+ = -10$ and $q_{\max} = 10,000$, the implied mean, variance, skewness, and kurtosis of the stationary distribution for $\alpha = 0.8$ are, respectively, given by 1.01 (μ), 1.63 (σ^2), 0.26 ($\mathcal{S}$), and 3.14 ($\mathcal{K}$).

Comparing these results with the exact central moment values reported in Table 1 of De Gooijer (2022), we see that the quality of the approximation is good. By contrast, for $\alpha = 0.9$ ($m^- = -6$, $m^+ = 10$, $q_{max} = 9,000$) the results are less satisfactory with 1.96 (μ), 2.49 (σ^2), 0.23 ($\mathcal{S}$), and 3.00 ($\mathcal{K}$).

References

De Gooijer, J.G. (2022). The marginal distribution function of threshold-type processes with central symmetric innovations. *Statistics*, 56.

Hervé, L. and Ledoux, J. (2020). State-discretization of V-geometrically ergodic Markov chains and convergence to the stationary distribution. *Methodology and Computing in Applied Probability*, 22, 905–925.

Meyn, S.P. and Tweedie, R.L. (1993). *Markov Chains and Stochastic Stability*, Springer-Verlag, London.